Journal of Engineering Mathematics **28**: 55–71, 1994.

The non-linear evolution of modulated waves in a boundary layer

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Received 1 December 1992; accepted in revised form 13 September 1993

Abstract. The series of experiments by Schubauer and Skramstad (1948) provided the first experimental evidence of the role that the instability of Tollmien–Schlichting waves played in the transition of a zero pressure-gradient flat plate boundary layer. The initial experiments studied the oscillations in the boundary layer excited by the freestream fluctuations. This was only possible after the background disturbances in the wind tunnel had been reduced to a very low level. The background wind tunnel environment excited a broad band of amplitude modulated disturbance waves that grew as they propagated downstream, eventually leading to the formation of turbulent spots. Further experiments used artificial two-dimensional harmonic excitation to produce regular wavetrains that could be directly compared with linear theory. Unfortunately, two-dimensional harmonic excitation of this type has also been used in many of the subsequent nonlinear transition investigations; the modulation of the disturbance waves, essential in nonlinear studies, has been largely ignored. Gaster and Grant (1975) used a short duration acoustic pulse to excite the boundary layer and found that the modulated wavepacket that was created admitted bursts of high frequency oscillations. These occurred at amplitudes that were insufficient to generate non-linear behaviour in purely periodic wavetrains. Gaster (1980) suggested that the modulation of the wavepacket played an important role in the non-linear region of transition.

This investigation used computer generated deterministic white noise to excite the boundary layer on a flat plate through a single buried loudspeaker. This type of excitation produced amplitude modulated T–S waves at some point downstream from the source. By repeatedly exciting the boundary layer with the same white noise sequence it was possible to map the entire flow-field with a single hot-wire probe and so study the evolution of the modulated wavetrains and the eventual development of turbulent spots. The modulated wavetrains were found to grow initially according to linear theory. But downstream, departures from the linear pattern were observed at isolated time instants. The amplitude of the irregular portions of the signal increased rapidly with downstream distance until bursts of oscillations of frequencies five or six times the basic T–S frequency were observed. These regions developed even higher frequency bursts until a turbulent spot could be considered to have formed. Excitation signals of various amplitudes with different phase relations between the spectral components were used in these experiments. It was found that the phases between the Fourier components played an important role in the highly non-linear behaviour that is the precursor to a turbulent spot. Novel signal processing techniques, such as the wavelet transform and Singular Value Decomposition were used to investigate the fine structure and the propagation characteristics of the high frequency disturbances.

1. Introduction

Observations of the laminar/turbulent transition process have shown that there are many different paths to a fully developed self-sustaining turbulent flow. Even in studies covering a single flow configuration the pattern of behaviour may not be unique; it may well vary from one test facility to another. In the case of the boundary layer on a flat plate, to be discussed in the present paper, we find that the character of the excitation of the initially weak wave system dictates how the waves evolve as they propagate downstream, and therefore determines the mechanics of their eventual breakdown to turbulence. The initial disturbances in the boundary layer undergoing natural transition are generated by turbulence in the external free-stream flow. The intensity and spectral content of the excitation field will vary from one test facility to another, and will almost certainly be quite different from that encountered in fight. The problem of interpreting experimental observations of the transition process has been additionally confused by the large number of experiments studying transition in flows excited artificially by a periodic two-dimensional wavemaker. Such experiments have been invaluable in demonstrating the power of both linear and non-linear models of wave evolution, but these experiments often ignore the essential influence of the extraneous disturbances that cause the wave patterns to become modulated and three-dimensional before breaking down into a turbulent spot. These approaches also ignore the fact that the naturally occurring highly modulated waves break down through a somewhat different process from that followed by regular periodic waves.

Naturally excited waves detected by hot-wire anemometers in flight experiments, or in a regular type of wind tunnel of moderate turbulence, exhibit a broad-band character. The selective amplification of certain frequency bands causes the velocity fluctuations to develop into a deeply modulated waveform containing packets, or bursts, of Tollmien–Schlichting waves. Wavetrains of this type evolve quite differently from regular harmonically excited waves when the amplitudes are large enough for non-linear effects to be significant. The modulated wavetrains break down and form turbulent spots at a fraction of the amplitudes of the purely periodic wavetrains. This type of disturbance therefore warrants further study to gain more insight into the transition of boundary layer flows in both flight and wind tunnel conditions.

Here computer generated white-noise sequences have been used to excite the boundary layer on a flat plate through a single buried loud-speaker. This provided a point source disturbance at an upstream location that created a wedge of instability waves. Some distance downstream of this source the oscillations recorded by a hot-wire anemometer were highly modulated and quite similar to those seen in signals from naturally excited situations. This excitation technique, using computer generated white noise, not only reproduced the type of wavetrain that we wished to study, but it did so in a controlled deterministic manner. One could therefore explore the structure of the evolving wavetrain by repeatedly exciting the flow with an identical noise sequence, whilst recording the velocity fluctuations at different stations in the flow.

The general pattern of the transition process of the broad-band wavy oscillations into turbulent spots did not depend strongly on the level of the initial excitation. Of course the more strongly excited waves broke into turbulence sooner than those weakly excited, but the mechanics were the same. The oscillations initially increased in amplitude with downstream distance in a manner consistent with linear theory. Each spectral element developed independently of the rest of the spectrum until a modulated, narrow-band, T–S wave emerged. Further amplification and frequency selection of the T–S waves continued to reduce the bandwidth of the signal. At some stage, however, departures from the linear pattern could be seen in the signals at isolated time instants while the remainder of the signal appeared to propagate unaltered. During these time periods the hot-wire records also showed some sample-to-sample variation. Slightly further downstream these localised regions of the time series exhibited more violent excursions, often containing regions of high frequency oscillation. These high frequency oscillations were not chaotic turbulence, but appeared to consist of a relatively narrow band of frequencies some 5 or 6 times that of the basic T–S wave. Further downstream oscillations of even greater frequencies were evident. At this stage the results of repeated excitation of the same noise sequence showed considerable variations from one another. Also, the fine scale oscillations within these time periods became less regular and appeared to be chaotic. This can be considered to be the point in the flow where spots were born. Although the details of the oscillations within the spot varied from sample to sample, the overall pattern of behaviour was repeated. This experimental approach therefore provided a good way of studying the formation of spots and of their subsequent downstream development.

Excitation signals of different amplitudes and phase dispositions between the components were used in these experiments. It was found that the phases between the Fourier components played a significant role in the formation of the non-linear behaviour that caused a spot to form. The local non-linear distortions of the velocity profile that were observed prior to the formation of any secondary high frequency oscillations were similar to those recorded in wave packets and appeared to be associated with the formation of 'hair-pin' vortices.

Novel signal processing techniques, wavelet transforms and Singular Value Decomposition (SVD), have been applied to the hot-wire records in order to focus on the fine scale structure. The high frequency oscillation of some five or six times the T–S waveband could readily be picked out by the wavelet transform approach, as could the very high frequency 'tertiary' burst that sometimes appeared before true turbulence developed. The SVD technique was used to detect the sharp edges of the packet and so enabled the leading and trailing edges of spots to be tracked as they propagated downstream.

2. Experimental arrangement

The experiments were carried out in the 3 ft.-square closed return low turbulence wind tunnel in the Engineering Department of Cambridge University. All the experimental results presented here were obtained at a free-stream velocity of approximately 17.5 m/s. At that speed the turbulence intensity was close to 0.01% within the frequency band 4 Hz and 4 kHz. The aluminium flat plate that generated the boundary layer was 12.7 mm thick and 1.68 m long and was fitted with a 9:1 super elliptic leading edge of degree 3. A zero pressure gradient over the first 1 m of the plate was set up on the working face by adjusting the two-section trailing edge flap. The combination of a long flap of length 0.45 m followed by a short trim tab gave considerable control over the pressure distribution. The flat plate was mounted vertically in the centre of the working section of the tunnel.

The laminar boundary layer was excited by a miniature loudspeaker embedded in the plate 0.2 m from the leading edge on the centre line. Flow disturbances were generated by the loudspeaker via a single hole of 0.5 mm diameter.

Flow measurements were made with a Dantec constant temperature hot-wire anemometer. The gold plated tungsten wire of 2.5 μm was set up to give a frequency response in excess of 10 kHz, as indicated by the square-wave technique.

The experiment was controlled by an Archimedes micro-computer through an IEEE interface to a commercial modular signal control unit. The computer controlled the stepper motors driving the three-dimensional probe traverse and the digital-to-analogue (D/A) converter feeding the speaker. A single time base was used to control both the D/A and the 12-bit signal acquisition analogue-to-digital (A/D) converter so that the processes were synchronised. The unsteady component of the hot-wire signal was preamplified and filtered before being sampled. The data was stored on magnetic tape and processing was performed on a Sun Spark station.

The technique of exciting the laminar boundary layer in a high quality low turbulence environment, to study the process of transition to turbulence, has been common since the experiments of Schubauer and Skramstad [1]. However, most of the investigators to date have followed Schubauer in using periodic excitations. The resulting disturbances in the flow bear little resemblance to those seen in experiments on naturally excited waves that one can observe in an ordinary wind tunnel environment. Isolated turbulent spots only appear once the motion has become irregular. The current experiments have been devised to create disturbances similar to those occurring in natural transition. This then enables one to study the creation and subsequent development of turbulent spots.

The spots being studied here arose from an excitation signal that was sufficiently weak to allow the initial disturbance waves to grow linearly before becoming large enough for the breakdown process to occur. It was found desirable to excite a wide band of frequencies and to allow the natural Tollmien–Schlichting filtering process to create the deeply modulated wavetrains of interest. A white noise excitation signal was generated by the computer for this purpose. In the construction of this spectrum, each frequency was assigned a random phase between 0 and 2π. The 32-Kb time series was obtained by an inverse FFT applied to this spectrum. Three different time series were created with identical spectral content but with different randomised phases. The spectrum of one of these time series is shown on Fig. 1.

2.2. Experimental procedure

The flow oscillations generated by the white-noise excitation were mapped out in both space and time by measuring the response of the hot-write probe while positioned at different stations downstream. Since the excitation time series was stored it could be used repeatedly with the probe positioned at different stations. In this way the entire spatio-temporal evolution of the disturbance flow field could be obtained, including the way in which the signal varied through the boundary layer and across the span.

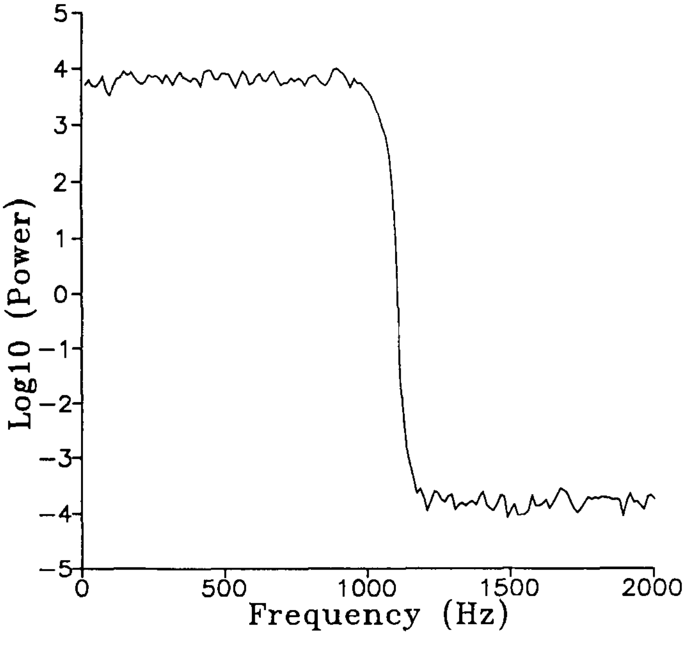

Fig. 1. Power spectrum of white noise excitation time series C.

Boundary layer traverses were made with the hot-wire to check that the mean flow velocity profiles were of the correct Blasius form. In fact they were indistinguishable from those calculated for a zero pressure gradient boundary layer. The most difficult part of this experiment was locating the relative position of the wall boundary to the probe with sufficient accuracy. The measured velocities were fitted to the theoretical profile to find the origin. The fits were extremely close and the position of the wall was not in any doubt. Having established this fact the various non-dimensional positions in the layer, denoted by $\eta = y\sqrt{U_0/2\nu x}$ were easily found. An automatic iterative scheme was used to locate the probe at a point in the Blasius profile that had a particular velocity ratio with respect to the free-stream and then the probe was repositioned at the required η location for data collection. All the results presented here were obtained where η was 1.1, which is near the inner maximum of the fluctuations of the boundary layer. The low frequency behaviour often seen in hot-wire records arises from the general heaving motion of the boundary layer. At a probe position just outside the boundary layer, where there is negligible mean profile gradient, this effect is no longer apparent. From previous work it has been found that the signals at this outer station give a good picture of any rapid fluctuations inside the layer, but do not respond to the violent non-linear excursions inside the layer.

Recordings of signals were made in the streamwise direction, X, from 800 mm from the leading edge to 1250 mm in steps of 50 mm, at intervals of 20 mm across the span, Z in the range ±120 mm from the centre line. At each measurement station 32 Kb of data were recorded at a 25 kHz sampling rate. The hot-wire anemometer signal was filtered in the range 4 Hz to 10 kHz. These records were repeated for the three white noise sequences generating in all 50 Mb of data that were stored on magnetic tape.

3. Data processing

The flow explored at these excitation levels spanned the region within which the non-linearity of the response was significant. Most of the records showed a relatively quiescent behaviour, interspersed by short duration large amplitude 'events'. During these events the signal showed oscillation frequencies many times that of the Tollmien–Schlichting waves that existed during the long intervals between them. This pattern of behaviour is illustrated in Fig. 2, which shows that the process contains very different time scales. It is this aspect of the signals that makes the processing difficult. Fourier methods are not able to resolve the very localised short duration high frequency elements – the spectrum of a whole record loses the high frequency components in the noise. To overcome this problem some sort of localised analysis is required. Both the wavelet transform and SVD provide this.

3.1. Singular value decomposition

The SVD technique (Broomhead and King [2]) extracts the mutually orthogonal shapes in the time series from the data set in a way that maximises an energy measure. Essentially, the method searches for energetic shapes commonly occurring in the signal. These vectors are believed to be associated with physical flow structures that occur in the transitional flow, in particular the instability waves (T–S waves) are selected as the dominant vector.

By projecting the signal onto the vector associated with the high frequency burst it is possible to produce a signal which defines the location of the bursts. Then by performing a

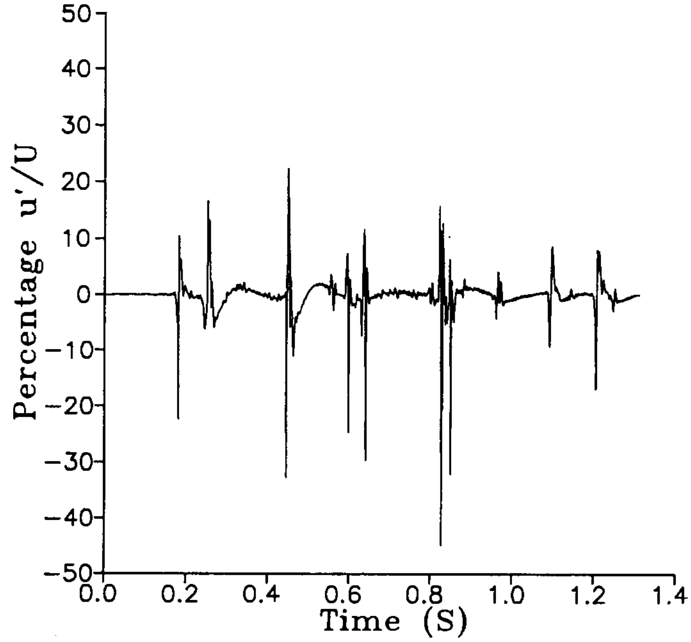

Fig. 2. Fluctuating streamwise velocity signal showing incipient spots.

Hilbert transform on the resulting signal, it is possible to estimate the proportion of the signal that contains high frequency components. Figure 3(a) shows a velocity record that contains a number of turbulent spots. Figure 3(b) shows the result after performing the projection and Hilbert transform, and Fig. 3(c) is the final signal from which an estimate of the turbulent intermittency can be made.

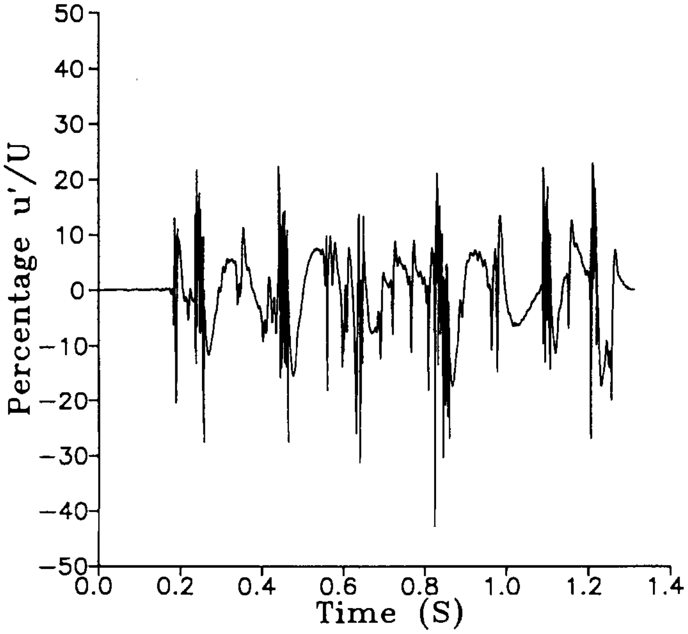

Fig. 3(a). The signal to be analysed.

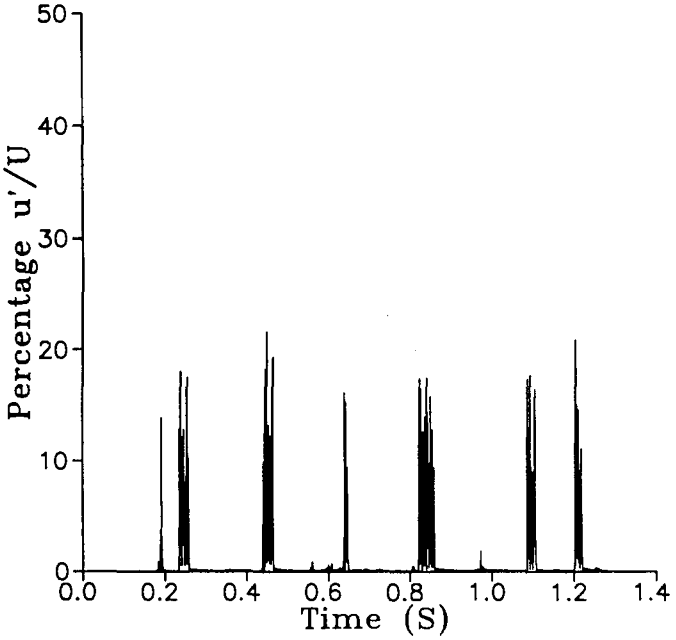

Fig. 3(b). The signal after primary processing.

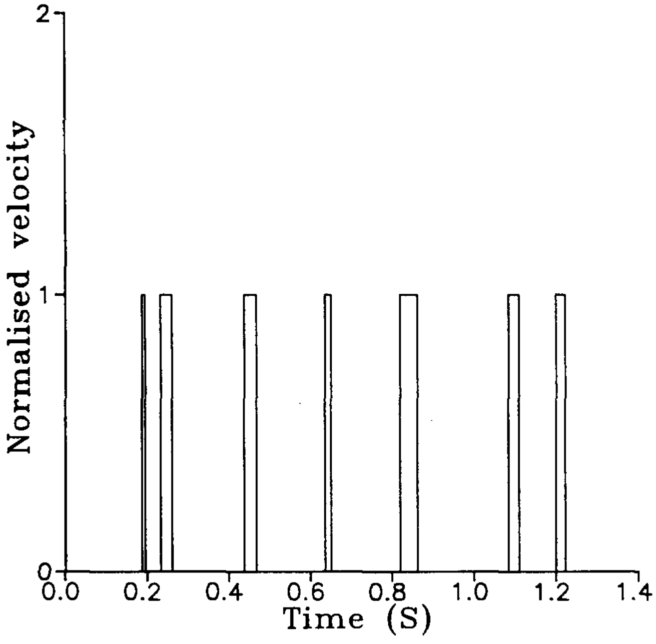

Fig. 3(c). The processed signal from which the intermittency is estimated.

3.2. Wavelet analysis

Shaikh and Gaster [3] described the use of the wavelet transform to characterise the high frequency bursts and so study their development into a turbulent spot. Farge [4] has given an overview of the wavelet transform and its many applications. Here, the wavelet transform is used to examine the frequency content within the bursts.

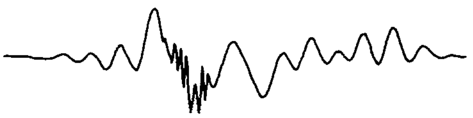

Fig. 4(a). The original time-series.

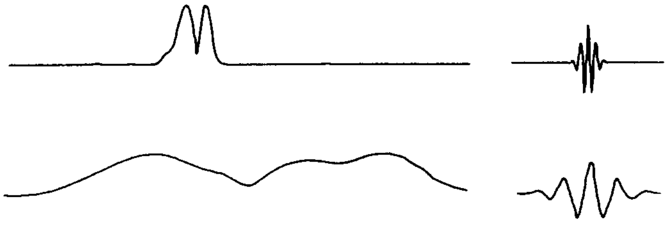

Fig. 4b. The magnitude of the wavelet coefficient at two scales and the corresponding analysing wavelets.

The wavelet transform produces a two-dimensional time/frequency map from a one-dimensional time series. This is done by projecting the time series onto a number of shapes, termed wavelets, which are all dilations of a basic shape, the analysing wavelet. The dilations of the analysing wavelet produce shapes of gradually increasing scale (or decreasing frequency), which detect like scales (or frequencies) in the signal to be analysed. Shaikh [5] compares the wavelet transform and the edge detection technique described in Section 3.1.

Figure 4(a) shows an experimental signal composed of a high frequency burst superimposed on an amplitude modulated T–S wavetrain. Figure 4(b) shows the way in which an analysing wavelet of two different scales detects the respective frequencies.

4. Discussion

4.1. Hot-wire records

Figure 5 shows the velocity fluctuations (u') in the stream direction obtained by a hot-wire anemometer 1.1 m from the leading edge and at 0.02 m off the centreline. This particular time series was created when the exciter was driven by white noise series (C) at its lowest amplitude. The resulting signal contained violent excursions, or spikes, separated by regions of inactivity. Although these spikes differ from one another in their detail and peak amplitude, there is some simularity in the overall pattern. Shaikh and Gaster [3] show that these spikes are correlated through the boundary layer and mark the occurrence of a short period during which the velocity profiles are inflectional. The three-dimensional structures associated with these profiles are the precursors of turbulent spots. In fact, one of the spikes in Fig. 5 (spot 1) contains a high frequency ripple or 'buzz' riding upon it. These high frequency oscillations rapidly develop into chaotic turbulence and so form a spot. Although the regions between the spikes or events appear to contain no significant disturbance, it turns out that there are highly modulating T–S waves present, albeit at a relatively low amplitude. They could be seen if the plotting scale were to be increased some 100 fold. The positions of the spikes are not obviously dependent on the amplitude of the exciting signal, but seem to depend also on the relative phases of the various spectral components.

Figure 6 shows how the signal evolves as it travels downstream a further 50 mm. The

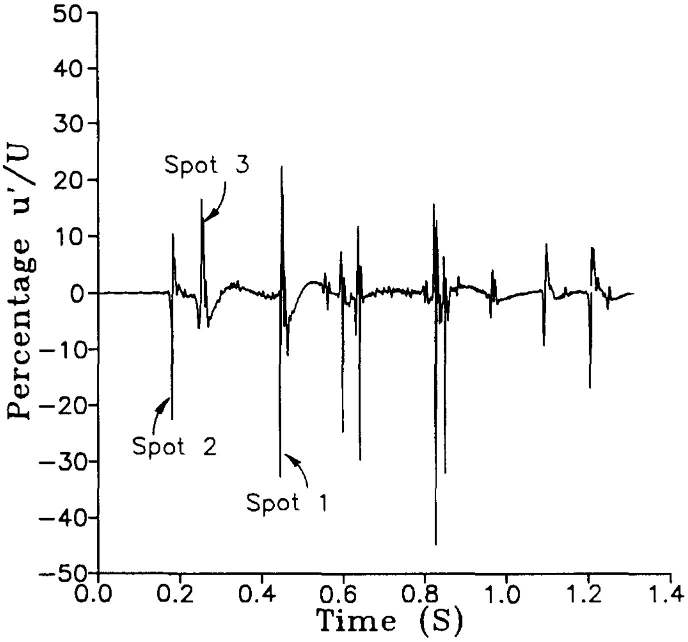

Fig. 5. Fluctuating streamwise velocity signal showing incipient spots, $X = 1.1$ m, $Z = 0.02$ m.

spikes occur at the same relative positions to those on Fig. 5 and show quite clearly that they are deterministic and are controlled by the input excitation. The amplitudes of the major excursions have increased over the 50 mm, and in addition, spots 2 and 3 now show some high frequency content. Spot 1 has grown in length and also shows an increase in the 'buzz'. In addition, new spikes have appeared from periods of the signal that were dormant at the upstream position.

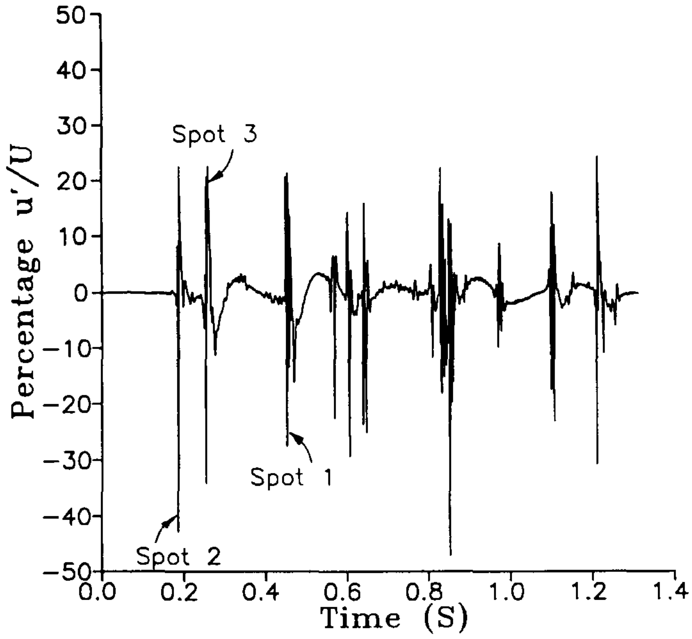

Fig. 6. Fluctuating streamwise velocity signal showing incipient spots, $X = 1.15$ m, $Z = 0.02$ m.

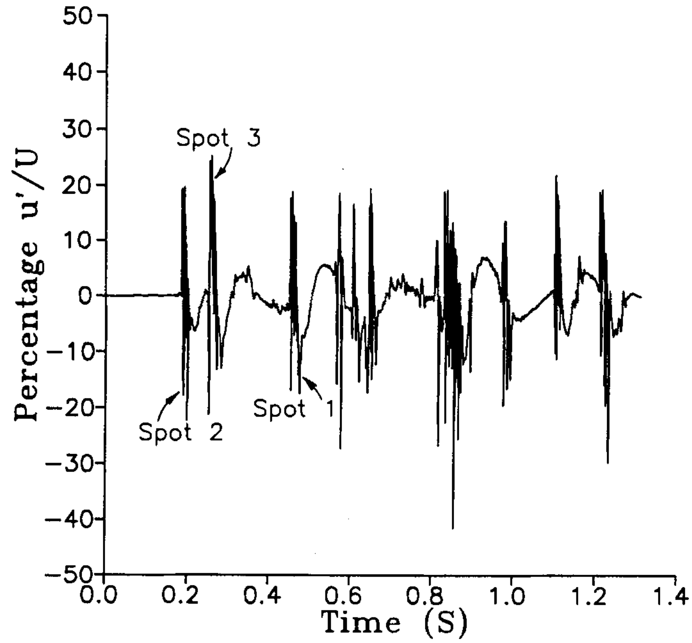

Fig. 7. Fluctuating streamwise velocity signal showing incipient spots, $X = 1.20$ m, $Z = 0.02$ m.

The progress of the signal as it propagates even further downstream is charted on Figs. 7 and 8 which show the time series at 100 mm and 150 mm downstream of the signal depicted on Fig. 5. Figure 7 again shows how some of the spots observed upstream have evolved. In particular, the previously observed weak excursions (young events) seem to grow in amplitude whereas the mature one appears to saturate and develop a greater degree of high

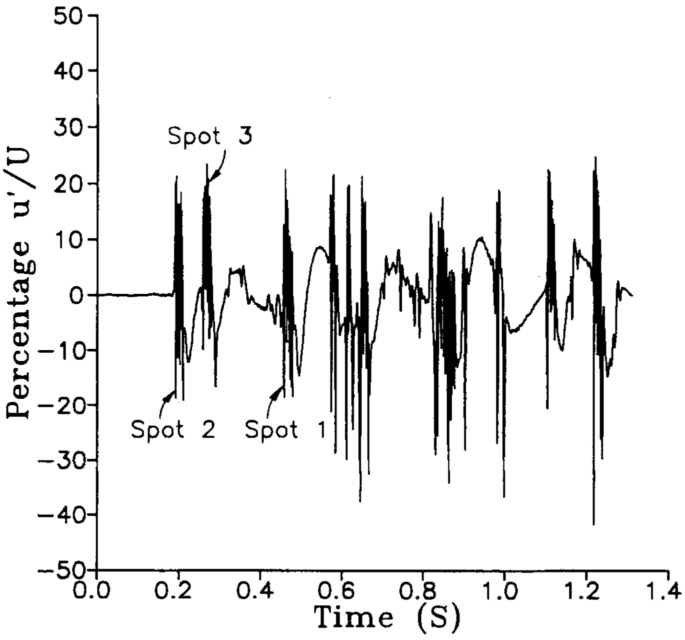

Fig. 8. Fluctuating streamwise velocity signal showing incipient spots, $X = 1.25$ m, $Z = 0.02$ m.

frequency oscillation. In fact, at this location ($X = 1.20$ m) the majority of the structures contain high frequency ripples, many of which appear also to be irregular. At $X = 1.25$ m, the hot-wire signals have developed into a series of turbulent spots (see Fig. 8). The turbulent spots continue to grow in extent with downstream propagation and eventually coalesce to form a fully turbulent flow field.

There are many ways of trying to resolve the high frequency, or turbulent portion of a signal, so that these regions can be delineated. It is possible to use the SVD approach discussed in Section 3.1 for this purpose. This is illustrated by the sequence of plots showing the development of one event as it travels downstream (see Figs. 9–12). Figure 9 shows that the spike has formed a region of higher frequency oscillations. The frequencies develop into a broader band form in Fig. 10, whilst Fig. 11 shows a very small fully turbulent spot containing a wide range of frequencies. Figure 12 shows a fully developed turbulent spot with discernible leading and trailing edges. These spots are very much smaller than those that have been studied previously through spark initiation at an upstream location. The small turbulent spots studied here have been produced in a deterministic manner and are similar to those occurring in natural transition.

4.2. Application of the wavelet transform

The previous section has given a qualitative description of the formation of a turbulent spot. Wavelet transforms can be used to give a more detailed description of the evolution in terms of both frequency and time. The signal displayed in Fig. 9 has been processed by a wavelet transform to provide the characterisation shown in Fig. 13(a). This type of display reveals how the various frequency components behave as functions of time. The burst of oscillations

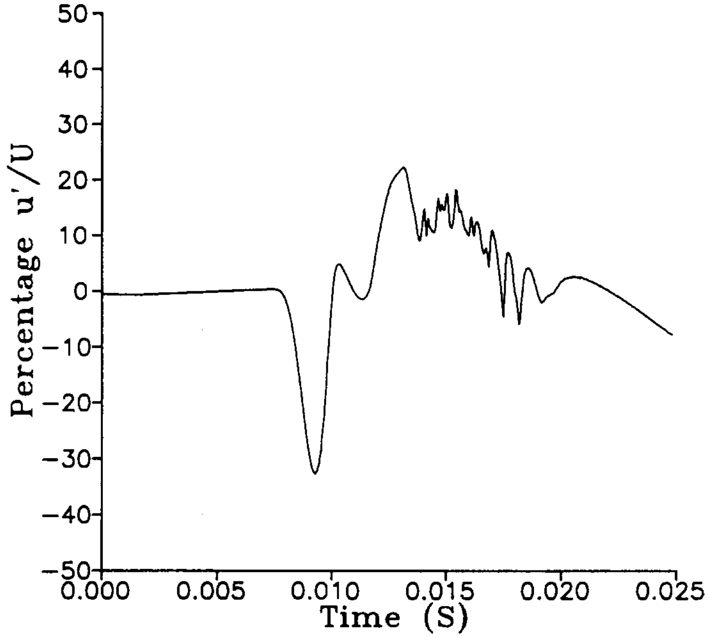

Fig. 9. Incipient spot 1, $X = 1.1$ m, $Z = 0.02$ m.

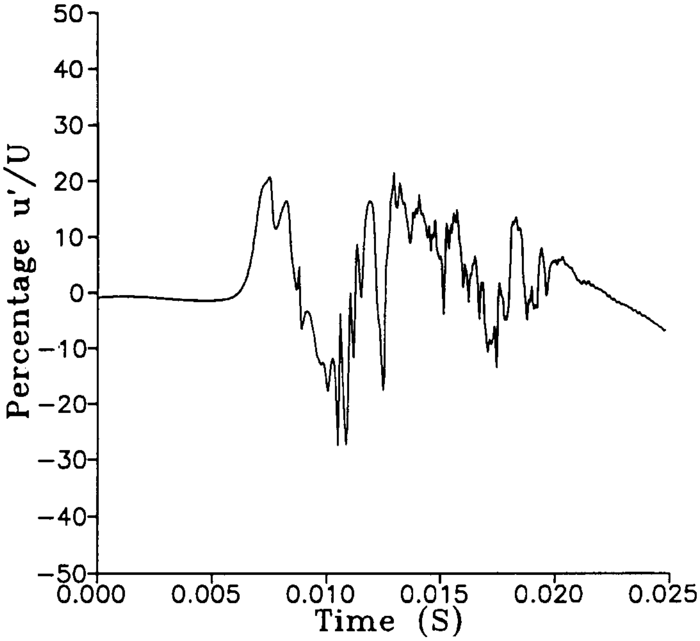

Fig. 10. Incipient spot 1, $X = 1.15$ m, $Z = 0.02$ m.

at roughly five times the fundamental T–S mode is clearly seen on both the wavelet plot and the original record, but the wavelet process provides a method of quantifying the behaviour.

This high frequency component has been previously observed in measurements on wave packets (Gaster [6]). In those experiments the distortion of the velocity profile was interpreted as the flow field produced by a horse-shoe vortex, the head moving outwards

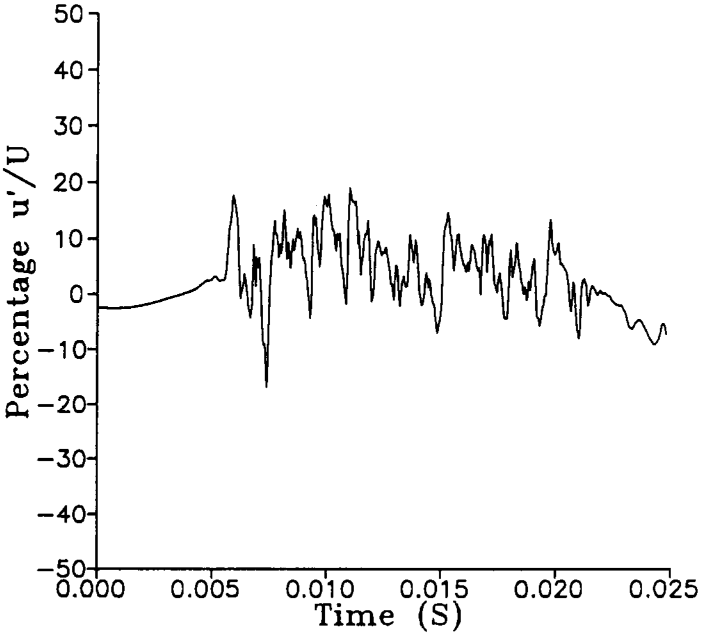

Fig. 11. Incipient spot 1, $X = 1.2$ m, $Z = 0.02$ m.

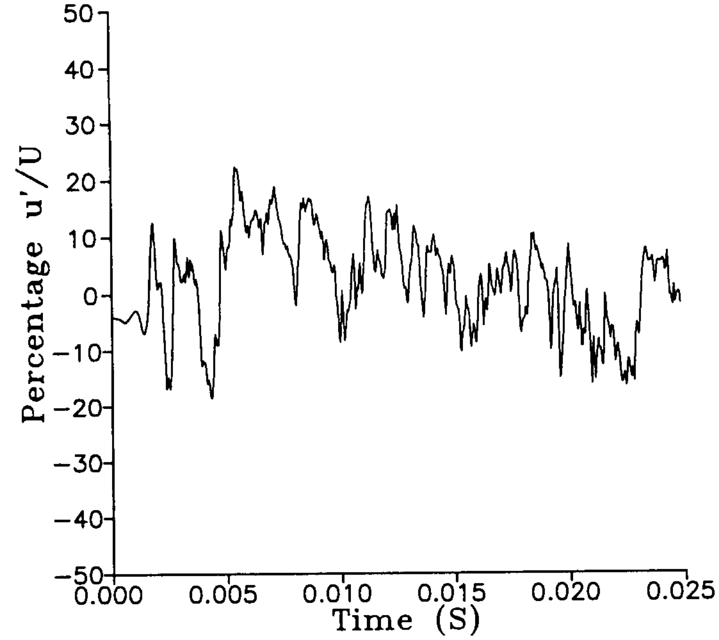

Fig. 12. Incipient spot 1, $X = 1.25$ m, $Z = 0.02$ m.

from the plate as the eddy propagated downstream. The resulting velocity profiles had very intense shear layers which exhibited secondary instability. The random free-stream excitation acted on this instability to create the high frequency burst. Numerical simulations of wave packet evolution, using the full Navier–Stokes equation, have also indicated the formation of 'hair-pin' structures (Fasel [7]).

The wavelet transforms of the record shown in Fig. 10 at a position 50 mm downstream from that of Fig. 9 is displayed in Fig. 13(b). The zone containing the secondary instability of some 5 times the T–S mode has developed to cover a longer portion of the record and shows some increase in bandwidth. The broadband nature of this high frequency component indicates that the flow is close to becoming turbulent.

Slightly further downstream the signal shows the occasional burst of a very much greater frequency (Fig. 13(c)). These 'tertiary' instabilities can only be properly picked out by the wavelet analysis which, in this case, clearly indicates some energy at a frequency roughly twenty times that of the fundamental T–S wave. At this stage of wave evolution the records did not repeat exactly from one realisation to the next. This random character was especially evident when examining the higher frequency elements.

4.3. Global characteristics of turbulent spots

The intermittency of a time series is defined as the proportion of the time that the signal is turbulent. The SVD method of signal processing was able to detect the leading and trailing edges of the turbulent zones and it was then a straightforward matter to estimate intermittency. Figures 14–17 show how this quantity develops in plan view for the excitation sequence (A) at four closely spaced levels of excitation ~1, 1.04, 1.08 and 1.12 (levels 1–4). This small range of excitation levels was chosen to ensure that the process of

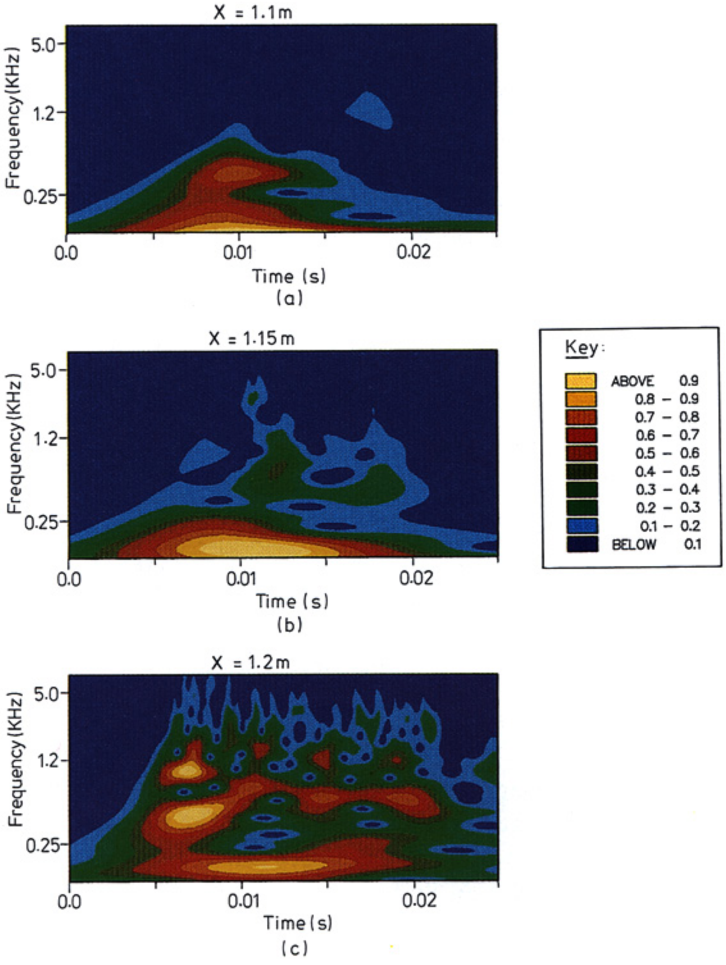

Fig. 13.

breakdown did not change radically through a different (bypass) process that would confuse the interpretation of the experiments. This series of plots show the region swept out by spots generated by the signal from th point source on the centre-line.

At the lowest amplitude of excitation, spots first occurred as very narrow periods of the time series at roughly 1 m downstream from the leading edge. The intermittency reached

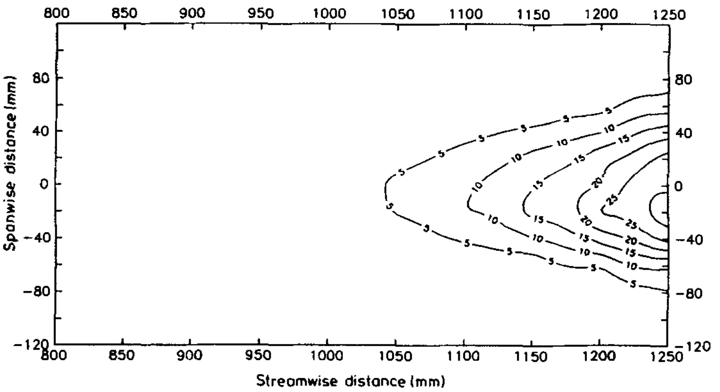

Fig. 14. Percentage intermittency – Series A, Amp 1.

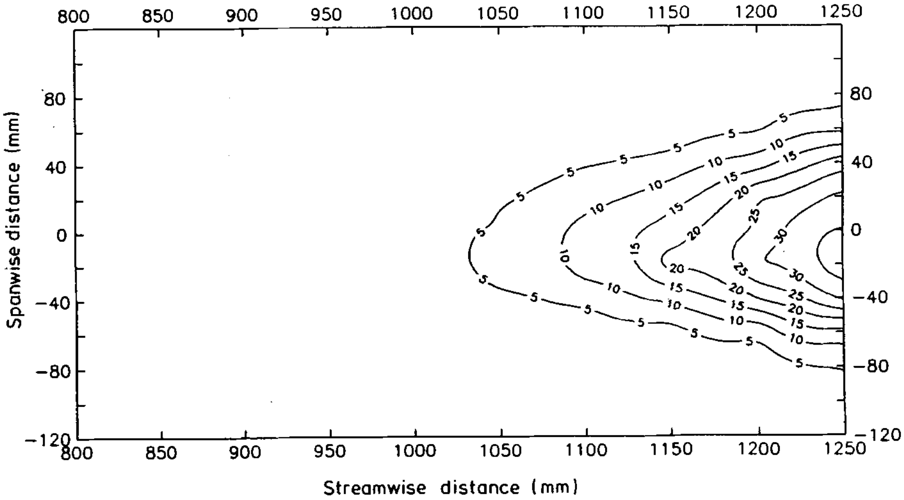

Fig. 15. Percentage intermittency – Series A, Amp 2.

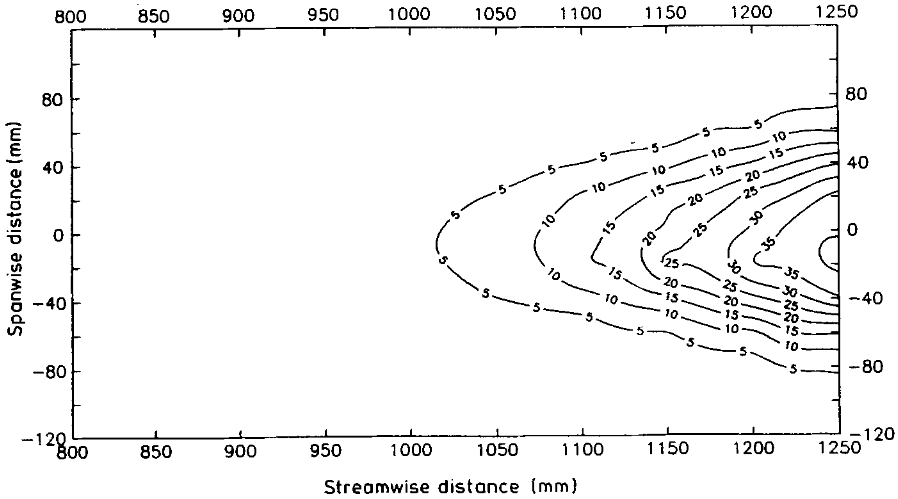

Fig. 16. Percentage intermittency – Series A, Amp 3.

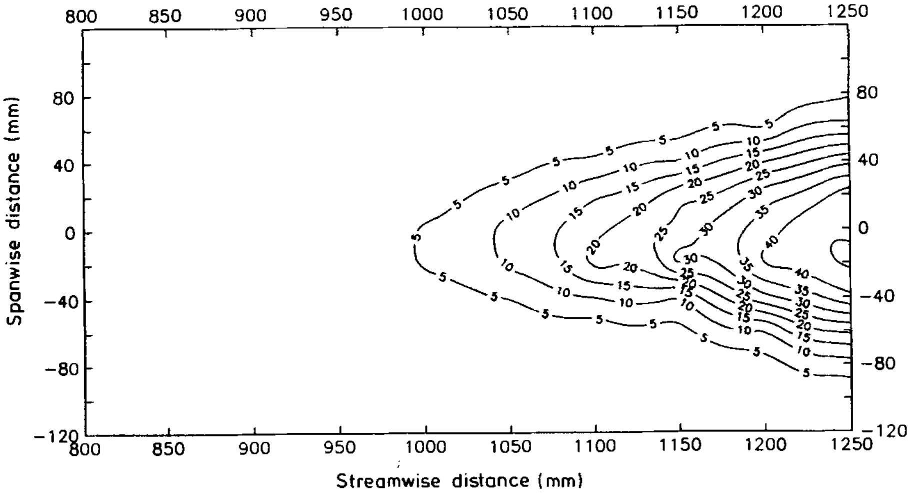

Fig. 17. Percentage intermittency – Series A, Amp 4.

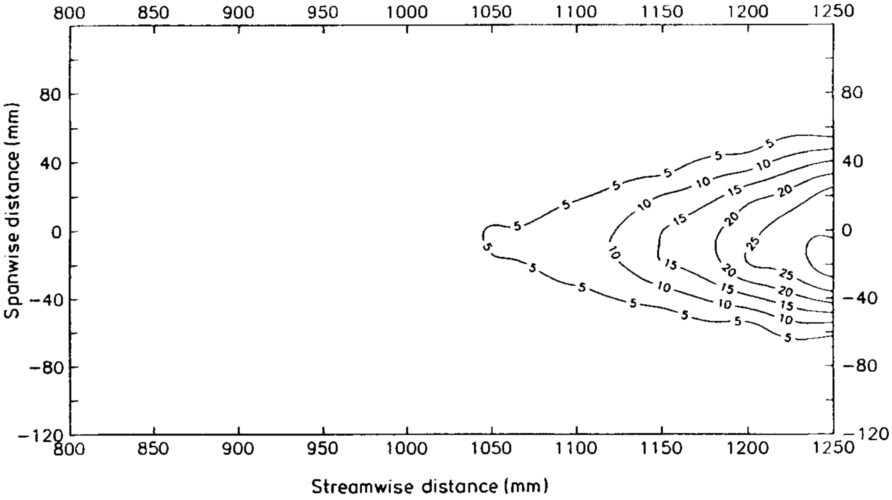

Fig. 18. Percentage intermittency – Series B, Amp 3.

35% at 1.25 m from the leading edge where the Reynolds number is 1.5×10^6. The contour pattern shows a slight asymmetry, presumably because the boundary layer was not precisely two-dimensional. Figures 15–17 show how this pattern of intermittency changed with excitation level. As perhaps might have been anticipated, the wedge moved upstream and the plots indicate that the spread in intermittency was more rapid at larger excitation.

These experiments were repeated with the two other white noise sequences of identical spectral content, but with different randomised phases. The data arising from excitation level 3 is shown for noise series B and C on Figs. 18 and 19, respectively. Figures 16, 18 and 19 show considerable differences and appear to have been produced by excitation of differing excitation amplitudes. Each record contained 32 Kb samples at an acquisition rate of 25 kHz giving a sampled time of 1.3 seconds, or roughly 260 T–S wave periods. Clearly, a very much longer time series is needed to form statistically representative estimates of intermittency, but the time to analyse them becomes prohibitive.

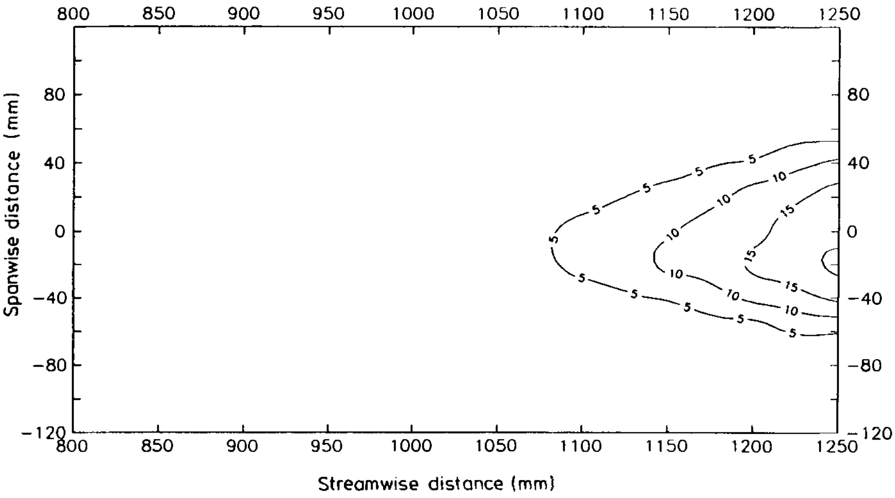

Fig. 19. Percentage intermittency – Series C, Amp 3.

5. Conclusions

The excitations provided by computer generated band limited sequences have been shown to create flow disturbances in the boundary layer that are similar to those that occur during the natural process of transition. The wavy disturbances develop as a deeply modulated Tollmien–Schlichting wavetrain that breaks down into small isolated turbulent spots at particular points in space and time. The process is controlled by the input excitation and is therefore deterministic in-so-far as the locations of the spots are concerned for any given sequence. This method of excitation therefore provides a useful way of studying spot formation and growth.

Local methods of signal processing involving SVD and wavelet transforms have been shown to be capable of focusing attention onto specific aspects of the non-linear bursts. In particular, the cascade of instabilities that eventually lead to chaotic turbulence can be seen in the transform.

The cause of the violent motions that manifest themselves on the signal traces as spikes are as yet unclear. From other studies it is thought that they are the result of a hairpin vortex passing over the wire. It is conjectured that the non-linear terms in the equations impose an additional stress field that induces this violent local eruption, presumably through a finite time Euler type of instability that is eventually moderated by viscosity.

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